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Algebraic K-Theory

Semester project

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Contents

Introduction

Algebraic K-theory is a branch of algebra dealing with linear algebra over a general ring A instead of over a field. It associates to any ring A a sequence of abelian groups $K_i(A)$. The first three of these, $K_0(A), K_1(A)$ and $K_2(A)$, can be described in concrete terms ; the others are rather mysterious. For instance, $K_0(A)$ is the group defined by the isomorphic classes of projectives modules over A and $K_1(A)$ is the abelianisation of the colimit of $GL_n(A)$. In the same way, $K_2(A)$ can be described in terms of generators and relations.

K-theory as an independent discipline is a fairly new subject, only about 50 years old. However, special cases of K-groups occur in almost all areas of mathematics, and particular examples of what we now call K_0 were among the earliest studied examples of abelian groups. We can still say that the letter K has been chosen from the German word Klasse.

Algebraic K-theory plays an important role in many subjects, especially number theory, algebraic topology and algebraic geometry. For instance, the class group of a number field K is essentially $K_0(O_K)$, where O_K is the ring of integers. Some formulas in operator theory, involving determinants, are best understood in terms of algebraic K-theory.

In this document, I will briefly intruduce the definitions of the K-theory groups. There is two parts : the first one is based on the book of John Milnor, *Introduction to algebraic K-theory*, and will give an algebraic definition of $K_0(A), K_1(A), K_2(A)$ and some properties of them ; the second one is based on Allen Hatcher's *Algebraic Topology* and will present the topological construction of the space that will define the higher K-theory groups.

Chapter 1

Preliminaries

We assume that the notions of ring, module, homomorphism between rings, etc. are known. In all the document, a ring will be an associative ring with $1 \neq 0$. An homomorphism ϕ between two rings will always satisfy $\phi(1) = 1$. Moreover, $\mathbb{N}$ will designe the set $\{0, 1, 2, ...\}$ and $\mathbb{N}^*$ will be $\mathbb{N} \setminus \{0\}$.

For all this chapter we fix a ring A. For any A-module M and for any subset $B \subseteq M$, we recall that $\langle B \rangle$ is the intersection of all the A-submodules of M having B as a subset. In fact we have

$$\langle B \rangle = \{\sum_{i=1}^{n} \lambda_i b_i \mid \lambda_i \in A, b_i \in B\}$$

Definition 1.1 *Let M be an A-module. A subset $B \subseteq M$ is called a system of generators of M if $\langle B \rangle = M$. In this case we say that B generates M.*

Definition 1.2 *An A-module M is called finitely generated if there is a subset $B \subseteq M$ which generates M and is finite.*

If one system of generators B has only one element, we say that M is cyclic.

Remark Generally there is more than one system of generators for an A-module M. In fact we can even have two sytems of generators which have not the same number of elements.

Example A is always a cyclic A-module. It is generated by 1.

Definition 1.3 *A basis B of an A-module M is a subset $B \subseteq M$ that generates M and is free, meaning that there are no relations between the elements of B in M.*

Definition 1.4 *An A-module L is called free if there is a basis B of L.*

Examples

1. The A-module A has $\{1\}$ as a basis and so is a free module.
2. If $A = K$ is a field, then a K-module is a K-vector space and so have a basis. In fact this result is true if A is a division ring.
3. The polynom ring $A[X]$, seen as an A-module, has $\{1, X, X^2, ...\}$ as a basis.
4. A^n is a free module over A with basis $\{e_i \mid 1 \leq i \leq n\}$, where e_i is the element $(0, ..., 0, 1, 0, ..., 0) \in A^n$ with the 1 at the i-th place.
5. The $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is a finitely generated module (even cyclic), but doesn't have any basis.

Proposition 1.5 *If L and L' are two free A-modules, then $L \oplus L'$ is a free A-module.*

Proof. If B and B' are basis for L and L' respectively, then it is clear that $B \times B'$ is a basis for $L \times L' \cong L \oplus L'$.

Proposition 1.6 *Every free and finitely generated A-module L is isomorphic to an A-module A^n, with $n \in \mathbb{N}$.*

Proof. Since L is free and finitely generated, there is a finite basis B for L. So we can write $B = \{b_1, ..., b_n\}$. We consider the map

$$\begin{aligned} \phi : A^n &\longrightarrow L \\ (x_1, ..., x_n) &\longmapsto \sum_{i=1}^{n} x_i b_i \end{aligned}$$

ϕ is well defined and is clearly an A-homomorphism. Moreover ϕ is injective because B is free and ϕ is onto L because B generates L. So ϕ is an A-isomorphism. Thus $L \cong A^n$.

Remark

1. Since the basis of a free A-module haven't the same cardinality in general, the $n \in \mathbb{N}$ in the proposition 1.6 isn't unique for all ring A.
2. We say that A has the property of the unique rank if the $n \in \mathbb{N}$ is uniquely determinated. Such ring satisfies

   $$A^n \cong A^m \Longleftrightarrow n = m$$

   Fields, division rings and principal rings have the property of the unique rank.

3. For a field or a division ring K, every finitely generated K-module is isomorphic to K^n, for a $n \in \mathbb{N}$. Moreover, the $n \in \mathbb{N}$ is unique, since K is a field.

Definition 1.7 *An A-module P is called projective if there exists an A-module Q so that $L := P \oplus Q$ is a free module over A.*

Remark In the case of the definition 1.7, we have that Q is also a projective module over A :

$$Q \oplus P \cong P \oplus Q = L$$

Examples

1. A free module L is always projective because $L \oplus 0 \cong L$ is free.

2. A projective module is always a submodule of a free module. Effectively, if P is a projective module, there is one Q so that $P \oplus Q$ is free. So $P \cong P \oplus 0 \subseteq P \oplus Q$ is a submodule of a free module.

3. The $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is not projective.

   In fact a free $\mathbb{Z}$-module is a direct sum of copy of $\mathbb{Z}$ (since proposition 1.6) and so is torsionless, i.e. there is no element x so that $nx = 0$ for an integer n. But $\mathbb{Z}/2\mathbb{Z}$ isn't torsionless and so cannot be submodule of a free $\mathbb{Z}$-module.

Proposition 1.8 *If P and Q are projective A-modules, then $P \oplus Q$ is also a projective module.*

Proof. Since P and Q are projective, there are A-modules M and N so that $P \oplus M$ and $Q \oplus N$ are free. By proposition 1.5, $P \oplus M \oplus Q \oplus N$ is free. But

$$P \oplus M \oplus Q \oplus N \cong P \oplus Q \oplus M \oplus N$$

and so $P \oplus Q$ is projective.

Chapter 2

The group K_0

2.1 Milnor's definition of K_0

Let A be a ring. To define $K_0(A)$ we consider the following equivalence relation. We say that two finitely projective A-modules P and Q are equivalent if and only if they are isomorphic, i.e. if there is an isomorphism of A-modules $P \longrightarrow Q$. This is clearly an equivalence relation.

We note $\overline{P}$ for the equivalence class of the projective A-module P and $Proj(A)$ for the set of all the equivalence classes.

Definition 2.1 *(Milnor) The projective module group $K_0(A)$ is the group defined by generators and relations as follows. For each elements $\overline{P}$ of $Proj(A)$ we take a generator $[P]$ and for each pair $[P], [Q]$ of generators we define the relation*

$$[P] + [Q] := [P \oplus Q]$$

Remark Since $P \oplus Q \cong Q \oplus P$ we have that $\overline{P \oplus Q} = \overline{Q \oplus P}$ and so $[P] + [Q] = [P \oplus Q] = [Q \oplus P] = [Q] + [P]$, meaning that $K_0(A)$ is an abelian group.

Proposition 2.2 *Every element of $K_0(A)$ can be expressed by the formal difference $[P_1] - [P_2]$ of two generators.*

Proof. Since $K_0(A)$ is generated by $\{[P] \mid \overline{P} \in Proj(A)\}$, then an element $[Q] \in K_0(A)$ can be written

$$[Q] = \sum_{i=1}^{n} (-1)^{k_i} [Q_i]$$

where $k_i \in \mathbb{N}$ and $\overline{Q_i} \in Proj(A)$. Up to a permutation of the indices we get

$$\begin{aligned}
[Q] &= \sum_{i=1}^{m}[Q_i] + \sum_{i=m+1}^{n} -[Q_i] \\
&= \sum_{i=1}^{m}[Q_i] - \sum_{i=m+1}^{n} [Q_i] \\
&= [\bigoplus_{i=1}^{m} Q_i] - [\bigoplus_{i=m+1}^{n} Q_i]
\end{aligned}$$

Defining $P_1 := \bigoplus_{i=1}^{m} Q_i$ and $P_2 := \bigoplus_{i=m+1}^{n} Q_i$ we conclude that $[Q] = [P_1] - [P_2]$.

Remark The group $K_0(A)$ can be defined more formally as a quotient of a free abelian group. Effectively, we form the free abelian group F generated by the set $Proj(A)$ and we take the quotient by the normal subgroup R spanned by all $\overline{P} + \overline{Q} - \overline{P \oplus Q}$, where $\overline{P}, \overline{Q} \in Proj(A)$. So we have

$$K_0(A) = F/R$$

(To see more about free groups, consult [2].)

Definition 2.3 *Two A-modules M and N are called stably isomorphic if there exists $r \in \mathbb{N}$ so that*

$$M \oplus A^r \cong N \oplus A^r$$

Proposition 2.4 *Two generators $[P]$ and $[Q]$ of $K_0(A)$ are equal if and only if P is stably isomorphic to Q.*

Proof. As we have seen in the remark above, we can write $K_0(A)$ as a quotient F/R where F is a free abelian group. First note that a sum $\overline{P_1} + ... + \overline{P_k}$ in F is equal to $\overline{Q_1} + ... + \overline{Q_k}$ if and only if

$$P_i \cong P_{\sigma(i)}, \quad \forall i = 1, ..., k$$

for some permutation σ of $\{1, ..., k\}$. If this is the case, then we have clearly

$$P_1 \oplus ... \oplus P_k \cong Q_1 \oplus ... \oplus Q_k$$

Now suppose that we have $[P] = [Q]$ and so $\overline{P} \equiv \overline{Q}$ mod R. Then this means that

$$\overline{P} - \overline{Q} = \sum_{i=1}^{n} \overline{P_i} + \overline{Q_i} - \overline{P_i \oplus Q_i}$$

which is equivalent to

$$\overline{P} + \sum_{i=1}^{n} \overline{P_i \oplus Q_i} = \overline{Q} + \sum_{i=1}^{n} \overline{P_i} + \sum_{i=1}^{n} \overline{Q_i}$$

for some $n \in \mathbb{N}$ and appropriate projective modules P_i, Q_i. Applying the begining of the proof we get

$$P \oplus (\sum_{i=1}^{n} P_i \oplus Q_i) \cong Q \oplus (\sum_{i=1}^{n} P_i \oplus \sum_{i=1}^{n} Q_i)$$

Defining $X := \sum_{i=1}^{n} P_i \oplus Q_i \cong \sum_{i=1}^{n} P_i \oplus \sum_{i=1}^{n} Q_i$, we get that $P \oplus X \cong Q \oplus X$. Since X is projective, we can choose an A-module Y so that $X \oplus Y$ is free. By the proposition 1.6, $X \oplus Y \cong A^r$, for some $r \in \mathbb{N}$. Then we obtain

$$\begin{aligned} P \oplus X \cong Q \oplus X &\Longrightarrow P \oplus X \oplus Y \cong Q \oplus X \oplus Y \\ &\Longrightarrow P \oplus A^r \cong Q \oplus A^r \end{aligned}$$

Hence P is stably isomorphic to Q.

Conversely if P is stably isomorphic to Q, then there exists $r \in \mathbb{N}$ so that $P \oplus A^r \cong Q \oplus A^r$. So we have $[P \oplus A^r] = [Q \oplus A^r]$, since A^r is clearly projective. But

$$[P \oplus A^r] = [Q \oplus A^r] \Rightarrow [P] + [A^r] = [Q] + [A^r] \Rightarrow [P] = [Q]$$

which concludes the proof.

Corollary 2.5 *Two elements $[P_1] - [P_2]$ and $[Q_1] - [Q_2]$ of $K_0(A)$ are equal if and only if $P_1 \oplus Q_2$ is stably isomorphic to $P_2 \oplus Q_1$.*

Proof. $[P_1] - [P_2] = [Q_1] - [Q_2] \iff [P_1] + [Q_2] = [P_2] + [Q_1] \iff [P_1 \oplus Q_2] = [P_2 \oplus Q_1]$ and then we can conclude by the preceeding proposition.

2.2 Grothendieck's construction of K_0

Definition 2.6 *A monoid is a set G with an associative law which has an identity element, noted 1_G.*

If the law is commutative, then we say that G is an abelian monoid. In this case we note + the law and 0_G the identity element.

Examples

1. Any group is a monoid ; any abelian group is an abelian monoid.
2. $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$ are abelian monoids.
3. $\mathbb{Z}$ with the usual multiplication is also an abelian monoid.
4. $Proj(A)$ with the operation $\overline{P} + \overline{Q} := \overline{P \oplus Q}$ is an abelian monoid.

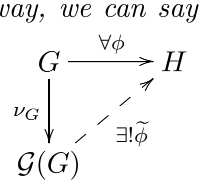
Definition 2.7 *Let $(G, \star)$ and $(H, \bullet)$ be monoids. An homomorphism of monoids is a map of sets*

$$\phi : G \longrightarrow H$$

so that $\phi(x \star y) = \phi(x) \bullet \phi(y)$, $\forall x, y \in G$, and that $\phi(1_G) = 1_H$.

Theorem 2.8 *Let G be an abelian monoid. Then there exists an abelian group $\mathcal{G}(G)$ and an homomorphism of monoids $\nu_G : G \longrightarrow \mathcal{G}(G)$ so that for all group H and for all homomorphism of monoids $\phi : G \longrightarrow H$, there exists one and only one homomorphism of groups $\widetilde{\phi} : \mathcal{G}(G) \longrightarrow H$ so that $\phi = \widetilde{\phi} \circ \nu_G$.*

In an other way, we can say that $(\mathcal{G}(G), \nu_G)$ satisfy the following universal property :



The pair $(\mathcal{G}(G), \nu_G)$ is called Grothendieck's construction of G.

Proof. On $G \times G$, we introduce the equivalence relation

$$(x, y) \sim (x', y') \Longleftrightarrow \exists z \in G \text{ so that } x' + y + z = x + y' + z$$

We note $[x, y]$ the equivalence class of (x, y) and $\mathcal{G}(G) := G \times G / \sim$. We define on $\mathcal{G}(G)$ the following operation :

$$[x, y] + [u, v] := [x + u, y + v]$$

This operation is associative, commutative and has $[x, x]$ as an identity element, $\forall x \in G$:

$$[x, x] + [u, v] = [x + u, x + v] = [u, v]$$

since $u + x + v = x + u + v$. Moreover, if $[x, y] \in \mathcal{G}(G)$, then we have the inverse element $-[x, y] := [y, x]$. Effectively,

$$[x, y] + [y, x] = [x + y, y + x] = 0 = [y + x, x + y] = [y, x] + [x, y]$$

Hence $\mathcal{G}(G)$ is an abelian group.

Now consider the map

$$\begin{aligned}\nu_G : G \longrightarrow & \mathcal{G}(G)\\ x \longmapsto & [x+x, x]\end{aligned}$$

Since $\nu_G(x+y) = [x+y+x+y, x+y] = [x+x+y+y, x+y] = [x+x, x] + [y+y, y] = \nu_G(x) + \nu_G(y)$ and $\nu_G(0) = [0,0] = 0$, ν_G is an homomorphism of monoids.

Let H be an abelian group and $\phi : G \longrightarrow H$ an homomorphism of monoids. We get

$$\begin{aligned}[x,y] &= [x,y] + [x+y, x+y] = [x+(x+y), y+(x+y)]\\ &= [x+x, x] + [y, y+y] = [x+x, x] - [y+y, y]\\ &= \nu_G(x) - \nu_G(y)\end{aligned}$$

So we must define $\widetilde{\phi} : \mathcal{G}(G) \longrightarrow H$ by

$$\widetilde{\phi}([x,y]) := \phi(x) - \phi(y)$$

which is well and uniquely defined and is an homomorphism of groups. Furthermore

$$\widetilde{\phi}(\nu_G(x)) = \widetilde{\phi}([x+x, x]) = \phi(x+x) - \phi(x) = \phi(x)$$

Proposition 2.9 *Let G be an abelian monoid. Then the Grothendieck's construction $(\mathcal{G}(G), \nu_G)$ is unique up to isomorphism.*

Proof. Let B be an abelian group and $\psi : G \longrightarrow B$ be an homomorphism of abelian monoids so that for every abelian group H and homomorphism of monoids $\phi : G \longrightarrow H$ there exists a group homomorphism $\overline{\phi} : B \longrightarrow H$ uniquely determinated so that $\phi = \overline{\phi} \circ \psi$.

Putting $H = \mathcal{G}(G)$ and $\phi = \nu_G$ we get that there exists a group homomorphism $\overline{\nu_G} : B \longrightarrow \mathcal{G}(G)$ so that $\nu_G = \overline{\nu_G} \circ \psi$. By a similar argument, using the universal property of $(\mathcal{G}(G), \nu_G)$, there exists a group homomorphism $\widetilde{\psi} : \mathcal{G}(G) \longrightarrow B$ so that $\psi = \widetilde{\psi} \circ \nu_G$. We obtain :

$$\begin{aligned}\overline{\nu_G} \circ \widetilde{\psi} \circ \nu_G &= \nu_G\\ \widetilde{\psi} \circ \overline{\nu_G} \circ \psi &= \psi\end{aligned}$$

We can immediately deduce that

$$\begin{aligned}\overline{\nu_G} \circ \widetilde{\psi} &= Id_{Im(\nu_G)}\\ \widetilde{\psi} \circ \overline{\nu_G} &= Id_{Im(\psi)}\end{aligned}$$

To end the proof we have just to show that $B = \text{Im}\ \psi$ and $\mathcal{G}(G) = \text{Im}\ \nu_G$. We consider the homomorphism $q : B \longrightarrow B/Im(\psi)$ given by the canonical projection. The two homomorphisms

$$\begin{aligned} \theta_1 : B &\longrightarrow B \times (B/Im(\psi)) \\ x &\longmapsto (x, q(x)) \end{aligned}$$

and

$$\begin{aligned} \theta_2 : B &\longrightarrow B \times (B/Im(\psi)) \\ x &\longmapsto (x, 0) \end{aligned}$$

make the following diagram commute :

$$\begin{array}{ccc} G & \xrightarrow{\psi \times 0} & B \times (B/Im(\psi)) \\ {\scriptstyle \psi}\downarrow & \nearrow{\scriptstyle \theta_i} & \\ B & & \end{array}$$

for $i = 1, 2$. By uniqueness we must have $\theta_1 = \theta_2$ and so $B = \text{Im}\ \psi$. A similar argument gives $\mathcal{G}(G) = \text{Im}\ \nu_G$.

Example If $G = \mathbb{N}$ with the addition, then $\mathcal{G}(\mathbb{N})$ is the group with all the elements of the form $n - m$ for $n, m \in \mathbb{N}$. So we obtain

$$\mathcal{G}(\mathbb{N}) \cong \mathbb{Z}$$

Definition 2.10 *If A is a ring, then $Proj(A)$ is an abelian monoid. So we can define*

$$K_0(A) := \mathcal{G}(Proj(A))$$

This definition is clearly the same as Milnor's.

Proposition 2.11 *If $A = K$ is a field or a division ring, then*

$$K_0(K) = \mathbb{Z}$$

Proof. As seen in chapter 1, every finitely generated $K-$module (and so every finitely generated projective $K-$module) is isomorphic to K^n, for one unique $n \in \mathbb{N}$. So we have an isomorphism

$$Proj(K) \cong \mathbb{N}$$

Since $\mathcal{G}(\mathbb{N}) \cong \mathbb{Z}$ we can conclude that $K_0(K) \cong \mathbb{Z}$.

Remark This result is true if A has the property of the unique rank. Thus

$$K_0(\mathbb{Z}) \cong \mathbb{Z}$$

Theorem 2.12 $K_0(-)$ *is a covariant functor from the category of rings and homomorphisms of rings to the category of abelian groups and homomorphisms of groups.*

Proof. Let A_1 and A_2 be two rings and let $\phi : A_1 \longrightarrow A_2$ be a ring homomorphism. Then ϕ induces a structure of A_1-module on A_2 as follows

$$a \cdot b := \phi(a)b, \quad \forall a \in A_1, \forall b \in A_2$$

Hence for every finitely projective module P over A_1 there exists a tensor product $A_2 \otimes_{A_1} P$. On this tensor product over A_1 we can put a structure of A_2-module defining $b' \cdot (b \otimes v) := (b'b) \otimes v$, $\forall b, b' \in A_2$, $\forall v \in P$. Then we can define

$$\begin{aligned} Proj(\phi) : Proj(A_1) &\longrightarrow Proj(A_2) \\ \overline{P} &\longmapsto \overline{A_2 \otimes_{A_1} P} \end{aligned}$$

We can verify that if A_3 is an other ring and if $\psi : A_2 \longrightarrow A_3$ is a ring homomorphism, we have $Proj(\psi \circ \phi) = Proj(\psi) \circ Proj(\phi)$ and $Proj(Id_{A_1}) = Id_{Proj(A_1)}$. Thus $Proj(-)$ is a covariant functor from the category of rings and homomorphisms of rings to the category of abelian monoids and homomorphisms of monoids.

Now let G_1 and G_2 be two abelian monoids and let $\psi : G_1 \longrightarrow G_2$ be an homomorphism of monoids. From the theorem 2.8 we have two Grothendieck's constructions $(\mathcal{G}(G_1), \nu_{G_1})$ and $(\mathcal{G}(G_2), \nu_{G_2})$ for G_1 and G_2 respectively. The monoid homomorphism $\nu_{G_2} \circ \psi : G_1 \longrightarrow \mathcal{G}(G_2)$ gives rise to an homomorphism of abelian groups

$$\mathcal{G}(\psi) : \mathcal{G}(G_1) \longrightarrow \mathcal{G}(G_2)$$

With this definition, $\mathcal{G}(-)$ is a covariant functor from the category of abelian monoids and homomorphisms of monoids to the category of abelian groups and homomorphisms between abelian groups.

Since $K_0(-) = \mathcal{G} \circ Proj(-)$, the theorem is proved.

Chapter 3

The group K_1

3.1 Whitehead's lemma and definition of K_1

Let A be a ring and $GL_n(A)$ denote the general linear group consisting of all $n \times n$ invertible matrices over A. For all $n \in \mathbb{N}^*$, we define the map

$$\begin{aligned} i_n : GL_n(A) &\longrightarrow GL_{n+1}(A) \\ B &\longmapsto \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Proposition 3.1 *The map i_n is an homomorphism of groups and is injective, $\forall n \in \mathbb{N}^*$.*

Proof. Let $B, C \in GL_n(A)$. From

$$i_n(I_n) = \begin{pmatrix} I_n & 0 \\ 0 & 1 \end{pmatrix} = I_{n+1}$$

and

$$i_n(BC) = \begin{pmatrix} BC & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} = i_n(B) i_n(C)$$

we have that i_n is an homomorphism of groups, $\forall n \in \mathbb{N}^*$. Clearly $i_n(B) = I_{n+1} \Longleftrightarrow B = I_n$ and so i_n is injective, $\forall n \in \mathbb{N}^*$.

Remark Since the proposition 3.1 we can see $GL_n(A)$ as a subgroup of $GL_{n+1}(A)$. Effectively, $GL_n(A) \cong Im(i_n)$ which is a subgroup of $GL_{n+1}(A)$.

Definition 3.2 *We define the general linear group of A by*

$$GL(A) := \bigcup_{n \in \mathbb{N}^*} GL_n(A)$$

Theorem 3.3 *$GL(A)$ is a group.*

Proof. Let $B, C, D \in GL(A)$. By definition of $GL(A)$, there exists $n \in \mathbb{N}^*$ so that $B, C, D \in GL_n(A)$. Since $GL_n(A)$ is a group, we get $(BC)D = B(CD)$ and the associativity of $GL(A)$.

The identity element of $GL(A)$ is the matrix I with 1 at every place on the diagonal and 0 everywhere else.

Let $B \in GL(A)$. There exists $n \in \mathbb{N}^*$ so that $B \in GL_n(A)$. Since $GL_n(A)$ is a group, B has an inverse matrix $B^{-1} \in GL_n(A)$. We obtain

$$\begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} BB^{-1} & 0 \\ 0 & I \end{pmatrix} = I$$

and so $GL(A)$ is a group.

Definition 3.4 *Let $n \in \mathbb{N}^*$. A matrix in $GL_n(A)$ is called elementary if it coincides with the identity matrix except for a single off-diagonal entry. We note $E_n(A)$ the subgroup of $GL_n(A)$ generated by all the elementary matrices.*

Remark Since $i_n(E_n(A)) \subset E_{n+1}(A)$, we can embed $E_n(A)$ in $E_{n+1}(A)$, $\forall n \in \mathbb{N}^*$.

Definition 3.5 *We define $E(A) := \bigcup_{n \in \mathbb{N}^*} E_n(A)$*

Remark For every $n \in \mathbb{N}^*$, $E_n(A)$ is a subgroup of $GL_n(A)$. Since $GL_n(A)$ is a subgroup of $GL(A)$, we have that $E(A)$ is also a subgroup of $GL(A)$.

Lemma 3.6 *Let $n \in \mathbb{N}^*$ and $D \in GL_n(A)$. Then $\begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix} \in E_{2n}$.*

Proof. We note e_{ij}^{λ} the elementary matrix with $\lambda \in A$ at the (i,j)-th place, where $i \neq j$. If $i \neq k$ and $j \neq l$, then $e_{ij}^{\lambda} e_{kl}^{\mu}$ is a matrix with 1 on the diagonal, λ at the (i,j)-th place, μ at the (k,l)-th place and 0 everywhere else. Generalizing this we can write, for a matrix $B = (b_{ij}) \in GL_n(A)$:

$$\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} = \prod_{i=1}^{n} \prod_{j=n+1}^{2n} e_{ij}^{b_{i(j-n)}} \in E_{2n}(A)$$

and as the same

$$\begin{pmatrix} I_n & 0 \\ B & I_n \end{pmatrix} = \prod_{i=n+1}^{2n} \prod_{j=1}^{n} e_{ij}^{b_{(i-n)j}} \in E_{2n}(A)$$

Thus we get

$$\begin{pmatrix} 0 & -D \\ D^{-1} & 0 \end{pmatrix} = \begin{pmatrix} I_n & -D \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ D^{-1} & I_n \end{pmatrix} \begin{pmatrix} I_n & -D \\ 0 & I_n \end{pmatrix} \in E_{2n}(A)$$

and therefore

$$\begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix} = \begin{pmatrix} 0 & -D \\ D^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in E_{2n}(A)$$

Lemma 3.7 *(Whitehead) $E(A)$ is equal to the commutator subgroup of $GL(A)$:*

$$E(A) = [GL(A), GL(A)]$$

Proof. We can see that $e_{ij}^{\lambda} = [e_{ik}^{\lambda}, e_{kj}^{1}]$ for $i \neq j$ and $k \neq i, j$. So

$$E(A) \subseteq [E(A), E(A)] \subseteq [GL(A), GL(A)]$$

Let $B, C \in GL(A)$. By definition of $GL(A)$, there exists $n \in \mathbb{N}^*$ so that $B, C \in GL_n(A)$. We have

$$\begin{pmatrix} BCB^{-1}C^{-1} & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} BC & 0 \\ 0 & (BC)^{-1} \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} C^{-1} & 0 \\ 0 & C \end{pmatrix}$$

and so $\begin{pmatrix} BCB^{-1}C^{-1} & 0 \\ 0 & I_n \end{pmatrix} \in E_{2n}(A)$ by the lemma 3.6. Thus

$$[GL(A), GL(A)] \subseteq E(A)$$

which concludes the proof.

Definition 3.8 *(Whitehead) We define $K_1(A)$ by the quotient*

$$K_1(A) := GL(A)/E(A)$$

It comes from lemma 3.7 that $K_1(A)$ is a group since $E(A)$ is a normal subgroup of $GL(A)$, and that $K_1(A)$ is abelian since $E(A)$ is the commutator subgroup. In other words, $K_1(A)$ is the abelianisation of $GL(A)$.

3.2 Properties of K_1

Remark If a ring A is commutative, then the determinant operation is defined. If A^* is the multiplicative group consisting of all invertible elements of A, then we have a surjective map

$$\det : GL(A) \longrightarrow A^*$$

We denote by $SL(A)$ the kernel of this homomorphism. Since $A^* \cong GL_1(A)$, we can also see A^* as a subset of $GL(A)$. Clearly

$$A^* \subset GL(A) \xrightarrow{\det} A^*$$

is the identity map. So we have the short exact sequence

$$1 \longrightarrow SL(A) \longrightarrow GL(A) \xrightarrow{\det} A^* \longrightarrow 1$$

that is split exact.

Lemma 3.9 *Let* $1 \longrightarrow G_1 \xrightarrow{\phi} H \xrightarrow{\psi} G_2 \longrightarrow 1$ *be a short exact sequence of groups that is split exact. Then*

$$H \cong G_1 \oplus G_2$$

Proof. By definition of split exact, there is a section $s : G_2 \longrightarrow H$ so that $\psi \circ s = Id_{G_2}$. Consider the following short exact sequence :

$$1 \longrightarrow G_1 \xrightarrow{\iota} G_1 \oplus G_2 \xrightarrow{\pi} G_2 \longrightarrow 1$$

where ι is the inclusion $x \mapsto (x, 1)$ and π is the projection $(x, y) \mapsto y$. We define

$$\begin{aligned} \alpha : G_1 \oplus G_2 &\longrightarrow H \\ (x, y) &\longmapsto \phi(x)s(y) \end{aligned}$$

Since $\operatorname{Im} \phi = \ker \psi$, we get that $\psi \circ \alpha(x, y) = \psi(\phi(x)s(y)) = \psi(\phi(x))\psi(s(y)) = y$ and so the following diagram commutes :

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_1 & \xrightarrow{\iota} & G_1 \oplus G_2 & \xrightarrow{\pi} & G_2 & \longrightarrow & 1 \\ & & \| & & \downarrow \alpha & & \| & & \\ 1 & \longrightarrow & G_1 & \xrightarrow{\phi} & H & \xrightarrow{\psi} & G_2 & \longrightarrow & 1 \end{array}$$

By the five lemma, α is an isomorphism.

Remark A short exact sequence

$$1 \longrightarrow G \xrightarrow{\phi} H \xrightarrow{\psi} F \longrightarrow 1$$

where F is a free abelian group, always splits. In fact, the section is defined by choosing a basis for F and elements in H that are sent by ψ on the basis elements. Then we extend by linearity and since there is no relation in F, this is well defined.

Proposition 3.10 *Let A be a ring. Then*

$$K_1(A) \cong A^* \oplus (SL(A)/E(A))$$

Proof. Since the lemma 3.9 and the remark which precedes it, we get that

$$\begin{aligned} \alpha : A^* \oplus SL(A) &\longrightarrow GL(A) \\ (a, B) &\longmapsto a \cdot B \end{aligned}$$

is an isomorphism (where a, B are seen in $GL(A)$ and $a \cdot B$ is given by the matricial multiplication). We consider now the following homomorphisms :

$$\begin{aligned} E(A) &\longrightarrow A^* \oplus SL(A) \\ B &\longmapsto (1, B) \end{aligned} \qquad \begin{aligned} A^* \oplus SL(A) &\longrightarrow A^* \oplus (SL(A)/E(A)) \\ (a, B) &\longmapsto (a, q(B)) \end{aligned}$$

where $q : GL(A) \longrightarrow GL(A)/E(A)$ is the canonical projection. Then we get a short exact sequence

$$1 \longrightarrow E(A) \longrightarrow A^* \oplus SL(A) \longrightarrow A^* \oplus (SL(A)/E(A)) \longrightarrow 1$$

Defining $\beta : A^* \oplus (SL(A)/E(A)) \longrightarrow K_1(A)$ by $\beta(a, q(B)) = q(a \cdot B)$, we get a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E(A) & \longrightarrow & GL(A) & \longrightarrow & GL(A)/E(A) & \longrightarrow & 1 \\ & & \| & & \downarrow \alpha & & \downarrow \beta & & \\ 1 & \longrightarrow & E(A) & \longrightarrow & A^* \oplus SL(A) & \longrightarrow & A^* \oplus SL(A)/E(A) & \longrightarrow & 1 \end{array}$$

By the five lemma we can conclude that β is an isomorphism and so that

$$K_1(A) = GL(A)/E(A) \cong A^* \oplus (SL(A)/E(A))$$

Proposition 3.11 *If $A = K$ is a field or a division ring, then*

$$K_1(K) \cong K^*$$

Proof. Since the preceeding proposition, it is enough to prove that $SL(K) = E(K)$. For an elementary matrix $E \in E(K)$ it is clear that $\det(E) = 1$ and so $E \in SL(K)$. Thus $E(K) \subseteq SL(K)$. To show the converse we use classical linear algebra. To make things more clear, we will note $e_{ij}(\lambda)$ for e_{ij}^{λ}.

Let $B = (b_{ij}) \in GL_n(K)$. Since B is invertible, the first column of B can't consist entirely of zeroes, i.e. there exists $i \in \mathbb{N}$, $1 \leq i \leq n$, so that $b_{i1} \neq 0$. If $i = 1$, this is fine. If not,

$$e_{1i}(1)e_{i1}(-1)e_{1i}(1)B$$

put b_{i1} in the $(1,1)$-position. So we can assume that $b_{11} \neq 0$. Adding $-b_{i1}b_{11}^{-1}$ times the first row to the $i-$th row for $i \neq 1$, i.e premultiplying B by

$$e_{n1}(-b_{n1}b_{11}^{-1}) \cdot ... \cdot e_{21}(-b_{21}b_{11}^{-1})$$

we can now kill all the other entries in the first column. This reduce B to the form

$$\begin{pmatrix} b_{11} & * \\ 0 & B_1 \end{pmatrix}$$

with B_1 an $(n-1) \times (n-1)$ matrix. Since $\det(B) = b_{11}\det(B_1)$, we have that B_1 is an invertible matrix. Repeating the same procedure by induction we get

$$EB = \begin{pmatrix} b_{11} & * & * & ... & * \\ 0 & b'_{22} & * & ... & * \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & b'_{nn} \end{pmatrix} =: B'$$

with $E \in E(K)$ and all diagonal elements different from 0.

Now premultipling B' by $e_{1n}(-b'_{1n}(b'_{nn})^{-1}) \cdot ... \cdot e_{n-1,n}(-b'_{n-1,n}(b'_{nn})^{-1})$, we kill all the entries in the last column except b'_{nn}. Continuing by induction, we can now obtain

$$E'B' = \begin{pmatrix} b_{11} & 0 & 0 & ... & 0 \\ 0 & b'_{22} & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & b'_{nn} \end{pmatrix} =: B''$$

with $E' \in E(K)$ and $\det(B'') = \det(E')\cdot\det(B') = \det(E')\cdot\det(E)\cdot\det(B) = \det(B)$.

Finally, we have to transform the diagonal matrix B'' into a diagonal matrix with at most one diagonal entry different from 1. Using lemma 3.6, for $a \in K^*$, we have that

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in E(A)$$

and so that

$$E_a^k := \begin{pmatrix} I_k & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \in E(A)$$

for all $k \in \mathbb{N}$. In consequence we get

$$E^0_{b'_{nn}...b'_{22}} \cdot ... \cdot E^{n-3}_{b'_{nn}b'_{n-1,n-1}} \cdot E^{n-2}_{b'_{nn}} \cdot B'' = \begin{pmatrix} b_{11}b'_{22}...b'_{nn} & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{pmatrix} =: D$$

and so $D = E''B''$ for a $E'' \in E(K)$.

Since $\det(D) = \det(B'') = \det(B)$, we have, if $B \in SL(K)$, that $\det(D) = 1$. But $\det(B) = b_{11}b'_{22}...b'_{nn}$ and so $b_{11}b'_{22}...b'_{nn} = 1$. This means that $D = I_n$ and so that $B = (E''E'E)^{-1} \in E(K)$. Thus we have proved that $SL(K) \subseteq E(K)$, and so we may conclude.

Remark We can show that if $A = \mathbb{Z}$, then $SL(\mathbb{Z}) = E(\mathbb{Z})$. Hence

$$K_1(\mathbb{Z}) \cong \mathbb{Z}^* = \{-1, 1\}$$

Theorem 3.12 $K_1(-)$ *is a covariant functor from the category of rings and homomorphisms of rings to the category of abelian groups and homomorphisms of groups.*

Proof. Let $\phi : A_1 \longrightarrow A_2$ be an homomorphism of rings. We define

$$\begin{aligned} GL(\phi) : GL(A_1) &\longrightarrow GL(A_2) \\ (b_{ij}) &\longmapsto (\phi(b_{ij})_{ij}) \end{aligned}$$

and thus $GL(-)$ is a covariant functor from the category of rings and ring homomorphisms to the category of groups and group homomorphisms.

Let G be a group. We denote G^{ab} for the abelianisation of G, that is $G^{ab} = G/[G, G]$. For a group homomorphism $\psi : G_1 \longrightarrow G_2$ we define

$$\begin{aligned} (\psi)^{ab} : (G_1)^{ab} &\longrightarrow (G_2)^{ab} \\ [g] &\longmapsto [\psi(g)] \end{aligned}$$

which is well defined, since

$$\psi(ghg^{-1}h^{-1}) = \psi(g)\psi(h)\psi(g)^{-1}\psi(h)^{-1} \in [G_2, G_2]$$

$\forall g, h \in G_1$. So we have $(-)^{ab}$ a covariant functor from the category of groups and homomorphisms of groups to the category of abelian groups and homomorphisms between abelian groups.

Then we can conclude, since $K_1(-) = (GL(-))^{ab}$.

Chapter 4

The group K_2

4.1 Definition of K_2

Let A be a ring. As in the preceeding chapter, let $e_{ij}^{\lambda} \in GL_n(A)$ denote the elementary matrix with entry λ in the i-th row and j-th column, where i and j can be any distinct integer between 1 and n and λ can be any ring element. We note that

$$e_{ij}^{\lambda} e_{ij}^{\mu} = e_{ij}^{\lambda+\mu}$$

Moreover we see that the commutator of two elementary matrices can be expressed as follows :

$$\begin{array}{lll} [e_{ij}^{\lambda}, e_{kl}^{\mu}] = & 1 & \text{if } j \neq k,\, i \neq l \\ [e_{ij}^{\lambda}, e_{kl}^{\mu}] = & e_{il}^{\lambda\mu} & \text{if } j = k,\, i \neq l \\ [e_{ij}^{\lambda}, e_{kl}^{\mu}] = & e_{kj}^{-\mu\lambda} & \text{if } j \neq k,\, i = l \end{array}$$

Definition 4.1 *Let $n \in \mathbb{N}, n \geq 3$. The Steinberg group $St_n(A)$ is the group defined by the quotient F_n/R_n where F_n is the free group generated by the symbols x_{ij}^{λ}, $1 \leq i, j \leq n$, $i \neq j$, $\lambda \in A$, and R_n is the smallest normal subgroup of F_n generated by the following elements :*

1. $x_{ij}^{\lambda} x_{ij}^{\mu} (x_{ij}^{\lambda+\mu})^{-1}$
2. $[x_{ij}^{\lambda}, x_{jl}^{\mu}](x_{il}^{\lambda\mu})^{-1}$ *for* $i \neq l$
3. $[x_{ij}^{\lambda}, x_{kl}^{\mu}]$ *for* $j \neq k$ *and* $i \neq l$

Remark Let $n \in \mathbb{N}, n \geq 3$, and $\lambda \in A$. The element $x_{ij}^{\lambda} \in F_n$ can be seen as an element of F_{n+1}. Since $R_n \subseteq R_{n+1}$ we have an homomorphism of groups

$$\begin{aligned} j_n : St_n(A) &\longrightarrow St_{n+1}(A) \\ x_{ij}^{\lambda} &\longmapsto x_{ij}^{\lambda} \end{aligned}$$

Moreover,

$$x_{ij}^{\lambda} \in \ker j_n \Longleftrightarrow j_n(x_{ij}^{\lambda}) \in R_{n+1} \Longrightarrow x_{ij}^{\lambda} \in R_{n+1} \Longrightarrow x_{ij}^{\lambda} \in R_n$$

since $0 \leq i, j \leq n$. So j_n is injective and we can embed $St_n(A)$ in $St_{n+1}(A)$.

Definition 4.2 *Because of the remark above we can form the group*

$$St(A) := \bigcup_{n \geq 3} St_n(A)$$

Remark The formula $\Phi_n(x_{ij}^{\lambda}) := e_{ij}^{\lambda}$ gives a well defined homomorphism

$$\Phi_n : St_n(A) \longrightarrow GL_n(A)$$

since each of the defining relations between generators of $St_n(A)$ maps into a valid identity between elementary matrices. The image $\Phi_n(St_n(A))$ is equal to the subgroup $E_n(A)$ generated by all elementary matrices of size $n \times n$.

Effectively, for every $e_{ij}^{\lambda} \in E_n(A)$, $\Phi_n(x_{ij}^{\lambda}) = e_{ij}^{\lambda}$ and conversely, for every $x_{ij}^{\lambda} \in St_n(A)$, $\Phi_n(x_{ij}^{\lambda}) = e_{ij}^{\lambda} \in E_n(A)$. So the generators of $E_n(A)$ are in bijection with generators of $St_n(A)$.

When we pass to the limit as $n \to \infty$, we obtain an homomorphism

$$\Phi : St(A) \longrightarrow GL(A)$$

with image $E(A) = [GL(A), GL(A)]$.

Definition 4.3 *The group $K_2(A)$ is defined as the kernel of the canonical homomorphism $\Phi : St(A) \longrightarrow GL(A)$.*

Proposition 4.4 *The sequence*

$$1 \longrightarrow K_2(A) \xrightarrow{\iota} St(A) \xrightarrow{\Phi} GL(A) \xrightarrow{q} K_1(A) \longrightarrow 1$$

is exact, where ι is the inclusion and q is the canonical projection.

Proof. Results immediately of the definition of $K_2(A)$ and of the fact that $\text{Im } \Phi = E(A)$.

Lemma 4.5 *Let $n \geq 3$ and let P_n denote the subgroup of $St(A)$ generated by elements $x_{1n}^{\mu}, x_{2n}^{\mu}, ..., x_{n-1,n}^{\mu}$ where μ ranges over A. Then each element of P_n can be written uniquely as a product*

$$x_{1n}^{\mu_1} x_{2n}^{\mu_2} ... x_{n-1,n}^{\mu_{n-1}}$$

Hence the canonical homomorphism Φ maps P_n isomorphically into the group $E(A)$.

Proof. Because of 3 in the definition 4.1, P_n is an abelian group. In consequence this is clear that every element of P_n can be written as a product $x_{1n}^{\mu_1} x_{2n}^{\mu_2} ... x_{n-1,n}^{\mu_{n-1}}$. The uniqueness comes from the fact that the elements 1 and 2 of the definition 4.1 don't belong to P_n.

Theorem 4.6 *The group $K_2(A)$ is the center of the Steinberg group $St(A)$.*

Proof. Let $B = (b_{ij}) \in GL_n(A)$. Since

$$B \cdot e_{kl}^1 = \begin{pmatrix} b_{11} & b_{12} & ... & b_{1,l-1} & b_{1l} + b_{1k} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2,l-1} & b_{2l} + b_{2k} & ... & b_{2n} \\ ... & ... & ... & ... & ... & ... & \\ b_{n1} & b_{n2} & ... & b_{n,l-1} & b_{nl} + b_{nk} & ... & b_{nn} \end{pmatrix}$$

and

$$e_{kl}^1 \cdot B = \begin{pmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ ... & ... & ... & ... \\ b_{k-1,1} & b_{k-1,2} & ... & b_{k-1,n} \\ b_{k1} + b_{l1} & b_{k2} + b_{l2} & ... & b_{kn} + b_{ln} \\ ... & ... & ... & ... \\ b_{n1} & b_{n2} & ... & b_{nn} \end{pmatrix}$$

we get that B commutes with e_{kl}^1 only if $b_{kl} = 0$ and $b_{kk} = b_{ll}$. In consequence we obtain that B commutes with every elementary matrix if and only if B is a diagonal matrix, with every diagonal entry equal to b_{11}. In particular, no element of $E_{n-1}(A)$ other that I_{n-1} belongs to the center of $E_n(A)$, for $n \geq 2$. Passing to the limit $n \to \infty$, it follows that $E(A)$ has a trivial center.

Now if c is in the center of $St(A)$, then $\Phi(c)$ is in the center of $E(A)$, which implies $\Phi(c) = I$ and so that

$$\text{center of } St(A) \subseteq K_2(A)$$

Conversely, suppose that $\Phi(y) = I$. Let $n \in \mathbb{N}$ so that $y \in St_{n-1}(A)$. Then we can write y with the generators x_{ij}^λ, $i, j < n$. Hence we get

$$x_{ij}^\lambda P_n x_{ij}^{-\lambda} \subseteq P_n$$

where P_n is defined as in the lemma 4.5. Effectively, $x_{ij}^\lambda x_{kn}^\mu x_{ij}^{-\lambda}$ is equal to x_{kn}^μ if $j \neq k$ and to $x_{in}^{\lambda\mu} x_{kn}^\mu$ if $j = k$. But $x_{kn}^\mu, x_{in}^{\lambda\mu} x_{kn}^\mu \in P_n$.

Since $y \in St_{n-1}(A)$, it follows that

$$yP_ny^{-1} \subseteq P_n$$

But $\Phi(y) = I$, thus $\Phi(ypy^{-1}) = \Phi(p)$, $\forall p \in P_n$. By the lemma 4.5, we get that $ypy^{-1} = p$ and so that y commutes with every element of P_n. Therefore y commutes with every generator x^{μ}_{kn}, $k < n$.

By an analogous argument we can show that y also commutes with every generator x^{μ}_{nl}, $l < n$. Hence y commutes with the commutator

$$[x^{\mu}_{kn}, x^{1}_{nl}] = x^{\mu}_{kl}$$

for all $k, l < n$, $k \neq l$. Since n can be as large as we want, y lies in the center of $St(A)$.

Corollary 4.7 *$K_2(A)$ is an abelian group.*

Theorem 4.8 *$K_2(-)$ is a covariant functor from the category of rings and homomorphisms of rings to the category of abelian groups and homomorphisms of groups.*

Proof. Let A_1 and A_2 be two rings and $\phi : A_1 \longrightarrow A_2$ be a ring homomorphism. We have seen in chapter 3 that ϕ induces an homomorphism $GL(\phi) : GL(A_1) \longrightarrow GL(A_2)$. Clearly this homomorphism satisfies $GL(\phi)(E(A_1)) \subseteq E(A_2)$. We define

$$\begin{aligned} \phi' : St(A_1) &\longrightarrow St(A_2) \\ x^{\lambda}_{ij} &\longmapsto x^{\phi(\lambda)}_{ij} \end{aligned}$$

and $K_2(\phi) := \phi'|_{K_2(A_1)}$. Then the following diagram commutes :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_2(A_1) & \longrightarrow & St(A_1) & \xrightarrow{\Phi_1} & E(A) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle GL(\phi)} & & \\ 0 & \longrightarrow & K_2(A_2) & \longrightarrow & St(A_2) & \xrightarrow{\Phi_2} & E(A_2) & \longrightarrow & 0 \end{array}$$

For $y \in K_2(A_1)$, we get by definition of $K_2(A_1)$ that $\Phi_1(y) = 0$. Therefore $(GL(\phi) \circ \Phi_1)(y) = 0$. Thus $(\Phi_2 \circ \phi')(y) = 0$ and so $\phi'(y) \in \ker \Phi_2 = K_2(A_2)$. Hence $K_2(\phi) : K_2(A_1) \longrightarrow K_2(A_2)$ is well defined, and make $K_2(-)$ a covariant functor.

4.2 Universal central extensions

Definition 4.9 *An extension of a group G is a pair (X, ϕ) consisting of a group X and an homomorphism of groups ϕ from X onto G.*

If $\ker(\phi)$ is a subset of the center of X we say that (X, ϕ) is a central extension.

Definition 4.10 *A central extension (X, ϕ) of a group G splits if it admits a section, that is an homomorphism $s : G \longrightarrow X$ so that $\phi \circ s = Id_G$.*

Proposition 4.11 *If a central extension (X, ϕ) of a group G splits then $X \cong G \times \ker \phi$.*

Proof. Since (X, ϕ) is a split extension of G we have a split short exact sequence

$$1 \longrightarrow \ker \phi \longrightarrow X \xrightarrow{\phi} G \longrightarrow 1$$

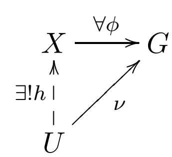
By the lemma 3.9, $X \cong G \times \ker \phi$.

Remark The splitting is given by

$$\begin{aligned} G \times \ker \phi &\longrightarrow X \\ (g, x) &\longmapsto s(g)x \end{aligned}$$

Definition 4.12 *A central extension (U, ν) of a group G is called universal if, for every central extension (X, ϕ) of G, there exists one and only one homomorphism from U to X over G. (That is, there exists one and only one homomorphism $h : U \longrightarrow X$ satisfying $\phi \circ h = \nu$.)*

We have then this commutative diagram :



Remark A universal central extension is always unique up to isomorphism over G.

Definition 4.13 *A group G is called perfect if it is equal to its commutator subgroup $[G, G]$.*

Examples

1. Since $[e^{\lambda}_{ik}, e^{1}_{kj}] = e^{\lambda}_{ij}$ if $i \neq j$, then $E(A) = [E(A), E(A)]$ and so $E(A)$ is perfect.

2. Since $[x^{\lambda}_{ik}, x^{1}_{kj}] = x^{\lambda}_{ij}$ if $i \neq j$, then $St(A) = [St(A), St(A)]$ and so $St(A)$ is perfect.

Proposition 4.14 *Let (Y, ψ) be a central extension of a group G. Then Y is perfect if and only if for all central extension (X, ϕ) of G there exists at most one homomorphism $Y \longrightarrow X$ over G.*

Proof. First suppose that Y is a perfect group and let (X, ϕ) be a central extension of G. Let f_1 and f_2 be homomorphisms from Y to X over G, meaning that $\phi \circ f_1 = \psi = \phi \circ f_2$. Hence we get, for all $y \in Y$,

$$\begin{aligned}\phi(f_2(y^{-1})f_1(y)) &= \phi(f_2(y^{-1}))\phi(f_1(y)) = \phi(f_2(y))^{-1}\phi(f_1(y)) \\ &= \psi(y)^{-1}\psi(y) = 1\end{aligned}$$

Then for any $y, z \in Y$ there exists $c, d \in \ker \phi$ so that

$$f_1(y) = f_2(y)c, \qquad f_1(z) = f_2(z)d$$

Since $\ker \phi$ is included in the center of X, then c, d are in the center of X. Therefore

$$\begin{aligned}f_1(yzy^{-1}z^{-1}) &= f_1(y)f_1(z)f_1(y)^{-1}f_1(z)^{-1} \\ &= f_2(y)cf_2(z)dc^{-1}f_2(y)^{-1}d^{-1}f_2(z)^{-1} \\ &= f_2(y)f_2(z)f_2(y)^{-1}f_2(z)^{-1} \\ &= f_2(yzy^{-1}z^{-1})\end{aligned}$$

and so $f_1 = f_2$, since Y is generated by commutators.

Conversely, suppose that Y isn't perfect. So there is a non-zero homomorphism $\alpha : Y \longrightarrow H$, where H is an abelian group. Let $(G \times H, \phi)$ be the central extention of G defined by $\phi(g, h) = g$. Clearly this extension is split, with section $s(g) = (g, 1)$. Setting

$$f_1(y) := (\psi(y), 1), \qquad f_2(y) := (\psi(y), \alpha(y))$$

we obtain two distinct homomorphisms from Y to $G \times H$ over G.

Lemma 4.15 *If (X, ϕ) is a central extension of a perfect group G, then the commutator subgroup $X' := [X, X]$ is perfect and maps onto G.*

Proof. Let $g_1, g_2 \in G$. Then there exists $x_1, x_2 \in X$ so that $\phi(x_1) = g_1$ and $\phi(x_2) = g_2$. So we get

$$\phi(x_1x_2x_1^{-1}x_2^{-1}) = g_1g_2g_1^{-1}g_2^{-1}$$

and then ϕ maps X' onto G, since G is generated by commutators.

Furthermore, for all $x \in X$ there exists $x' \in X'$ so that $\phi(x') = \phi(x)$. In consequence there exists $c \in \ker \phi$ (and so c is in the center of X) so that $x = x'c$. Then for $x_1, x_2 \in X$, there exists $x'_1, x'_2 \in X'$ and c_1, c_2 in the center of X so that $x_1 = x'_1c_1$ and $x_2 = x'_2c_2$. So we get

$$\begin{aligned}[x_1, x_2] &= x_1x_2x_1^{-1}x_2^{-1} = x'_1c_1x'_2c_2c_1^{-1}x_1'^{-1}c_2^{-1}x_2'^{-1} \\ &= x'_1x'_2x_1'^{-1}x_2'^{-1} = [x'_1, x'_2]\end{aligned}$$

and then $X' = [X', X']$.

Proposition 4.16 *A central extension (U, ν) of a group G is universal if and only if U is perfect and if every central extension of U splits.*

Proof. First suppose that U is perfect and every central extension of U splits. Let (X, ϕ) be a central extension of G and $U \times_G X$ be the subgroup of $U \times X$ consisting of all (u, x) with $\nu(u) = \phi(x)$. Then we define

$$\begin{aligned} \pi : U \times_G X &\longrightarrow U \\ (u, x) &\longmapsto u \end{aligned}$$

which is surjective since ϕ is onto G. Further, $\ker \pi = \{(0, x) \mid x \in \ker \phi\} = \{0\} \times \ker \phi$ commutes with every elements of $U \times_G X$, since (X, ϕ) is a central extension. Then $(U \times_G X, \pi)$ is a central extension of U, and by hypothesis has a section $s : U \longrightarrow U \times_G X$. Writing $s(u) = (s_1(u), s_2(u))$, we define

$$\begin{aligned} h : U &\longrightarrow X \\ u &\longmapsto s_2(u) \end{aligned}$$

Since $\pi \circ s = Id_U$, then $s_1(u) = u$. So $\phi(h(u)) = \phi(s_2(u)) = \nu(s_1(u)) = \nu(u)$ by the definiton of $U \times_G X$, and then h is an homomorphism from U to X over G. The uniqueness comes from the proposition 4.14, since U is perfect.

Conversely, suppose now that (U, ν) is a universal extension of G. From the proposition 4.14 it comes that U is perfect. Let (X, ϕ) be a central extension of U. We will prove that $(X, \nu \circ \phi)$ is a central extension of G.

Let $x_0 \in \ker(\nu \circ \phi)$. Then $\phi(x_0) \in \ker \nu$ and therefore $\phi(x_0)$ belongs to the center of U, since (X, ϕ) is central. Thus we get $\phi(x) = \phi(x_0)\phi(x_0^{-1})\phi(x) = \phi(x_0)\phi(x)\phi(x_0^{-1})$ and then there is an homomorphism from X to X over U defined as follows :

$$\begin{aligned} f : X &\longrightarrow X \\ x &\longmapsto x_0 x x_0^{-1} \end{aligned}$$

It comes from lemma 4.15 that the commutator subgroup X' is perfect and then from the proposition 4.14 that the homomorphism $f|_{X'} : X' \longrightarrow X'$ over U is the identity. Thus x_0 commutes with every elements of X'. But U is perfect and so, by lemma 4.15, there exists $x' \in X'$ so that $\phi(x') = \phi(x_0)$ and therefore $x_0 = x'c$ for a $c \in \ker \phi$. Since the extension is central, it follows that x_0 commutes with every $x \in X$. Thus $(X, \nu \circ \phi)$ is a central extension of G.

Since (U, ν) is universal, there exists an homomorphism $s : U \longrightarrow X$ over G. So $\phi \circ s$ gives an homomorphism from U to U over G, hence equals to the identity by proposition 4.14. Thus s is a section of (X, ϕ).

Lemma 4.17 *Let G be a group and $u, v, w \in G$. then*

1. $[u, v] = [v, u]^{-1}$
2. $[u, v][u, w] = [u, vw][v, [w, u]]$
3. $[u, [v, w]][v, [w, u]][w, [u, v]] \equiv 1 \ mod\ G''$

where $G'' := [[G, G], [G, G]]$ is the second commutator subgroup.

Proof. 1. $[u, v] = uvu^{-1}v^{-1} = (vuv^{-1}u^{-1})^{-1} = [v, u]^{-1}$.

2.
$$\begin{aligned} [u, vw][v, [w, u]] &= uvwu^{-1}w^{-1}v^{-1}vwuw^{-1}u^{-1}v^{-1}uwu^{-1}w^{-1} \\ &= uvu^{-1}v^{-1}uwu^{-1}w^{-1} \\ &= [u, v][u, w] \end{aligned}$$

3. By the first parts, we get that

$$\begin{aligned} [v, [w, u]] &= [u, vw]^{-1}[u, v][u, w] \\ &= [vw, u][u, v][u, w] \end{aligned}$$

Hence

$$\begin{aligned} &[u, [v, w]][v, [w, u]][w, [u, v]] = \\ &\qquad = [uv, w][w, u][w, v][vw, u][u, v][u, w][wu, v][v, w][v, u] \\ &\qquad \equiv [uv, w][vw, u][wu, v][w, u][w, v][u, v][u, w][v, w][v, u] \bmod G'' \\ &\qquad \equiv [uv, w][wu, v][vw, u] \bmod G'' \\ &\qquad \equiv uvwv^{-1}u^{-1}w^{-1}wuvu^{-1}w^{-1}v^{-1}vwuw^{-1}v^{-1}u^{-1} \bmod G'' \\ &\qquad \equiv uvww^{-1}v^{-1}u^{-1} \bmod G'' \\ &\qquad \equiv 1 \bmod G'' \end{aligned}$$

Theorem 4.18 *The Steinberg group $St(A)$ is actually the universal central extension of $E(A)$.*

Proof. Let $n \in \mathbb{N}$ so that $n \geq 5$. First we consider a central extension

$$1 \longrightarrow C \longrightarrow Y \xrightarrow{\phi} St_n(A) \longrightarrow 1$$

Given $x, x' \in St_n(A)$ we take $y \in \phi^{-1}(x)$ and $y' \in \phi^{-1}(x')$. We see that the commutator $[y, y']$ does not depend on the choice of y and y'. Effectively, let $z \in \phi^{-1}(x)$. Then we get

$$\phi(y^{-1}z) = \phi(y)^{-1}\phi(z) = x^{-1}x = 1$$

So we can choose $c \in \ker(\phi)$ so that $z = yc$ and, by a similar argument, $c' \in \ker \phi$ so that $z' = y'c'$. Since the extension is central we have that c and c' are in the center of Y and so

$$[z, z'] = [yc, y'c'] = ycy'c'(yc)^{-1}(y'c')^{-1} = yy'y^{-1}y'^{-1} = [y, y']$$

Now let x^1_{hi}, x^μ_{jk} be generators of $St_n(A)$. We suppose that i, j, k, h are distinct. Since $n \geq 5$ we can choose an $l \leq n$ distinct of i, j, k and h. Choosing

$$y \in \phi^{-1}(x^1_{hl}), \quad y' \in \phi^{-1}(x^1_{li}), \quad w \in \phi^{-1}(x^\mu_{jk})$$

we have that $[y, y'] \in \phi^{-1}(x^1_{hi})$ by 2 in definition 4.1. By the relation 3 we get that $[x^1_{hl}, x^\mu_{jk}] = 1$ and so that $[y, w] \in C$. As the same $[y', w] \in C$. This means that y and y' commute with w up to a central element and then that $[y, y']$ commutes with w. Thus we obtain

$$[\phi^{-1}(x^1_{hi}), \phi^{-1}(x^\mu_{jk})] = [[y, y'], w] = 1$$

Now choose $u \in \phi^{-1}(x^1_{hi})$ and $v \in \phi^{-1}(x^\lambda_{ij})$. Then $[u, w] = 1$. Further, if G is the subgroup of Y generated by u, v and w, then it follows from the relation 3 in the definition 4.1 that the commutator subgroup $G' = [G, G]$ is generated by elements in $\phi^{-1}(x^\lambda_{hj}), \phi^{-1}(x^{\lambda\mu}_{ik})$ and $\phi^{-1}(x^{\lambda\mu}_{hk})$. Then the second commutator subgroup $G'' = [G', G']$ is trivial. Therefore, by lemma 4.17,

$$[u, [v, w]] = [[u, v], w][[w, u], v] = [[u, v], w][1, w] = [[u, v], w]$$

and so that $[\phi^{-1}(x^\lambda_{hj}), \phi^{-1}(x^\mu_{jk})] = [\phi^{-1}(x^1_{hi}), \phi^{-1}(x^{\lambda\mu}_{ik})]$. Taking $\lambda = 1$, we obtain

$$[\phi^{-1}(x^1_{hj}), \phi^{-1}(x^\mu_{jk})] = [\phi^{-1}(x^1_{hi}), \phi^{-1}(x^\mu_{ik})]$$

and so the element

$$s^\mu_{hk} := [\phi^{-1}(x^\lambda_{hj}), \phi^{-1}(x^\mu_{jk})]$$

does not depend on the choice of j. Now it remains us to prove that these elements s^μ_{hk} satisfy the three Steinberg relations in definition 4.1. Then we will have that the correspondence $x^\mu_{hk} \longmapsto s^\mu_{hk}$ gives a well defined homomorphism from $St_n(A)$ to Y and that it is a section for

$$1 \longrightarrow C \longrightarrow Y \xrightarrow{\phi} St_n(A) \longrightarrow 1$$

Then every central extension of $St_n(A)$ splits and, passing to the limit when $n \to \infty$, every central extension of $St(A)$ splits. Thus we will be able to conclude from the fact that $St(A)$ is perfect and with the proposition 4.16.

Since $s^\mu_{hk} \in \phi^{-1}(x^\mu_{hk})$, we have the relation

$$[s^\lambda_{hj}, s^\mu_{jk}] = s^{\lambda\mu}_{hk}$$

for h, j, k distinct. Let $u \in \phi^{-1}(x^1_{hj})$, $v \in \phi^{-1}(x^\lambda_{jk})$ and $w \in \phi^{-1}(x^\mu_{jk})$. From the relation 2 in lemma 4.17, we get that

$$s^\lambda_{hk} s^\mu_{hk} = [u, v][u, w] = [u, vw][v, [w, u]]$$

But $[u, vw] = [\phi^{-1}(x_{hj}^1), \phi^{-1}(x_{jk}^{\lambda+\mu})] = s_{hk}^{\lambda+\mu}$ and $[v, [w, u]] = [v, [u, w]^{-1}] = [\phi^{-1}(x_{jk}^{\lambda}), \phi^{-1}(x_{hk}^{-\mu})] = 1$. So we obtain

$$s_{hk}^{\lambda} s_{hk}^{\mu} = s_{hk}^{\lambda+\mu}$$

Finally, we have from the first part of the proof that $[\phi^{-1}(x_{hi}^1), \phi^{-1}(x_{jk}^{\mu})] = 1$ and so the three Steinberg relations are proved.

Chapter 5

Higher K-theory groups

For this chapter, we suppose known the notions of action, fundamental group, covering space, universal covering space, fibration and cofibration and the theorem of van Kampen.

5.1 The B-construction

Definition 5.1 *Let $n \in \mathbb{N}$. The standard n-simplex is the convex subset of $\mathbb{R}^{n+1}$ defined by*

$$\Delta^n := \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0\}$$

The points $e_k = (0, ..., 0, 1, 0, ..., 0)$, with the 1 at the k-th position, are called the vertices of the simplex.

The sets $f_k := \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0, t_k = 0\}$ are called the faces of the simplex.

Δ^n is oriented by the natural ordering of its vertices and any face spanned by a subset of the vertices inherits an orientation as a subset of the vertices of Δ^n. Hence each face is canonically isomorphic to Δ^{n-1}, preserving the ordering.

Examples

- For $n = 0$ we obtain the point 1 in $\mathbb{R}$.
- The standard 1-simplex is the oriented segment from $(1, 0)$ to $(0, 1)$ in $\mathbb{R}^2$.

- The standard 2-simplex is the triangle in $\mathbb{R}^3$ with vertices $e_0 = (1,0,0)$, $e_1 = (0,1,0)$ and $e_2 = (0,0,1)$. Its edges are the oriented segments $[e_0, e_1]$, $[e_1, e_2]$ and $[e_0, e_2]$.

- For $n = 3$, we obtain the tetrahedron seen in $\mathbb{R}^4$ with vertices $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$.

Definition 5.2 *A Δ-complex structure on a topological space X is a collection of continuous maps $\sigma_\alpha : \Delta^n_\alpha \longrightarrow X$, with n depending on the index α, so that :*

1. *The restriction $\sigma_\alpha|_{int(\Delta^n_\alpha)}$ is injective, and each point of X is the image of exactly one such restriction.*

2. *Each restriction of σ_α to a face of the n-simplex Δ^n_α is one of the maps $\sigma_\beta : \Delta^{n-1}_\beta \longrightarrow X$. Here we identify the faces of Δ^n_α with a $(n-1)$-simplex in the canonical way, preserving the ordering of the vertices.*

3. *A subset $A \subseteq X$ is open if and only if $\sigma^{-1}_\alpha(A)$ is open in Δ^n_α for every α.*

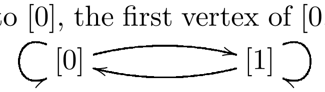
Remark With the condition 3, we can think of a Δ-complex as a quotient space of a collection of disjoint n-simplices, one for each α, the quotient space obtained by identifying each face of a Δ^n_α with the Δ^{n-1}_β corresponding to the restriction σ_β of σ_α to the face, as in condition 2.

Definition 5.3 *Let G be a group. For every $(n+1)$-tuple $(g_0, g_1, ..., g_n)$ of elements of G we write $[g_0, g_1, ..., g_n]$ for the n-simplex obtained by identifying g_i with e_i, $\forall i \in \mathbb{N}$, $i \leq n$.*

Definition 5.4 *Let G be a group. We note EG the Δ-complex whose n-simplices are all the ordered $(n+1)$-tuples $[g_0, g_1, ..., g_n]$ composed of elements of G and whose faces f_k are attached to the n-simplices $[g_0, ..., g_{k-1}, g_{k+1}, ...g_n]$.*

Example If $G = \mathbb{Z}/2 \cong \{0, 1\}$, then we construct EG as follows :

- First the 0-simplices are [0] and [1]

- The 1-simplices are $[0,0], [0,1], [1,0]$ and $[1,1]$. Then we attach the vertices of $[0,0]$ to $[0]$, the first vertex of $[0,1]$ to $[0]$ and the last to $[1]$, etc. We obtain 

- There is eight 2-simplices $[e_0, e_1, e_2]$. We attach the faces of $[e_0, e_1, e_2]$ to the 1-simplices $[e_0, e_1]$, $[e_0, e_2]$ and $[e_1, e_2]$.

- And so on...

Proposition 5.5 *G acts freely on EG, with action defined by*

$$\begin{aligned} g : EG &\longrightarrow EG \\ [g_0, g_1, ..., g_n] &\longmapsto [gg_0, gg_1, ..., gg_n] \end{aligned}$$

$\forall n \in \mathbb{N}$, $\forall g \in G$.

Proof. First we have to show that for $g, h \in G$ we have $g \circ h = gh$.

$$\begin{aligned} g(h([g_0, g_1, ..., g_n])) = g([hg_0, hg_1, ..., hg_n]) &= [ghg_0, ghg_1, ..., ghg_n] \\ &= (gh)([g_0, g_1, ..., g_n]) \end{aligned}$$

and so $g \circ h = gh$.

Furthermore we get that for every $g \in G$, g is a permutation of EG, i.e. g is a bijection. Effectively, g has an inverse g^{-1} in G. Then $g \circ g^{-1} = gg^{-1} = e = g^{-1}g = g^{-1} \circ g$ and $e = Id_{EG}$.

Now we have to prove that this action is free, meaning that there is no n-simplex $[g_0, g_1, ..., g_n] \in EG$ and no $g \in G$ other than e so that $g([g_0, g_1, ..., g_n]) = [gg_0, gg_1, ..., gg_n] = [g_0, g_1, ..., g_n]$. But

$$\begin{aligned} [gg_0, gg_1, ..., gg_n] = [g_0, g_1, ..., g_n] &\Longrightarrow gg_0 = g_0 \\ &\Longleftrightarrow gg_0g_0^{-1} = g_0g_0^{-1} \Longleftrightarrow g = e \end{aligned}$$

Definition 5.6 *Let G be a group. The B-construction of G is the orbit space $BG := EG/G$ of the action of the proposition 5.5.*

Lemma 5.7 *Let G be a group and $g \in G$. Then each $y \in EG$ has a neighborhood U so that $U \cap g(U) = \emptyset$ if $g \neq e$.*

Proof. The proof is based on the fact that G is acting freely and that an n-simplex is sent to a n-simplex by any element $g \in G$.

Proposition 5.8 *Let G be a group. The quotient map $q : EG \longrightarrow BG$ defined by $q(x) = Gx$ is a universal covering space.*

Proof. It is clear that q is surjective. Let $y \in Y$ and let U be a neighborhood of y as in lemma 5.7. Then we get that the sets $g(U), g \in G$, are disjoints and that

$$q^{-1}(q(U)) = \coprod_{g \in G} g(U)$$

But for every $g \in G$, the definition of the quotient topology gives that q is an homeomorphism from $g(U)$ to $q(U)$. Then $q : EG \longrightarrow BG$ is a covering space. Clearly, EG is path-connected. It remains us to prove that $\pi_1(EG) = 0$, i.e. EG is contractible.

Let $[g_0, ..., g_n] \in EG$ and $x \in [g_0, ..., g_n]$. Identifying $[g_0, ..., g_n]$ with Δ^n we can write $x = \sum_{i=0}^{n} t_i e_i$. Then we identify $\Delta^{n+1} = [e_0, ..., e_n, e_{n+1}]$ with $[g_0, ..., g_n, e]$ and we see x in Δ^{n+1} in the canonical way : $x = \sum_{i=0}^{n} t_i e_i + 0 e_{i+1}$. Thus we can define the homotopy

$$\begin{aligned} H : [0,1] \times \Delta^{n+1} &\longrightarrow \Delta^{n+1} \\ (s, x) &\longmapsto (1-s) \sum_{i=0}^{n} t_i e_i + s e_{n+1} \end{aligned}$$

Clearly $H(0, x) = x$ and $H(1, x) = [e]$. Then H is an homotopy from Id_{EG} to the projection $EG \longrightarrow [e]$. Then EG is contractible.

Proposition 5.9 *Let G be a group. Then $\pi_0(BG) = 0$ and $\pi_1(BG) \cong G$.*

Proof. Since proposition 5.8, q is a fibration and $\pi_0(EG) = 0 = \pi_1(EG)$. We note F for the fiber $q^{-1}(G[e])$. Since

$$g^{-1}([g]) = [g^{-1}g] = [e]$$

we have that $[g] \in G[e]$ and so that $G[g] = G[e]$. Thus $[g] \in F$, $\forall g \in G$. But it is clear that if $n \geq 1$, $q([g_0, g_1, ..., g_n])$ is a set of n-simplex and each of them cannot be equal to $g[e]$. Then we get

$$F = \{[g] \mid g \in G\} \cong G$$

In this case, the long exact sequence of the fibration q gives

$$0 = \pi_1(EG) \longrightarrow \pi_1(BG) \longrightarrow \pi_0(F) \cong \pi_0(G) \longrightarrow \pi_0(EG) = 0$$

Since G is a discreet space, $\pi_0(G) \cong G$ and so

$$\pi_1(BG) \cong G$$

Let $x, y \in BG$. Since q is surjective, there exists $x', y' \in EG$ so that $q(x') = x$ and $q(y') = y$. Since EG is path-connected, there is a path γ in EG from x' to y'. Then $q(\gamma)$ gives a path in BG from x to y. Then BG is path-connected and therefore $\pi_0(BG) = 0$.

5.2 Singular homology

In this section, we will briefly introduce the notion of singular homology, since we will need it in the next part to define the K-theory groups. Most of the properties won't be proved here.

Definition 5.10 *Let $n \in \mathbb{N}$. A singular n-simplex in a space X is a continuous map $\sigma : \Delta^n \longrightarrow X$.*

Definition 5.11 *Let X be a topological space and $n \in \mathbb{N}$. We denote by $C_n(X)$ the free abelian group with basis the set of singular n-simplices in X. We call an element of $C_n(X)$ a singular n-chain.*

Remark A singular n-chain is a finite formal sum $\sum_{i=1}^{k} n_i \sigma_i$ where $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \longrightarrow K$.

Definition 5.12 *Let X be a topological space and $n \in \mathbb{N}^*$. We define the boundary map $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ by the homomorphism given by formula*

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[e_0,\dots,e_{i-1},e_{i+1},\dots,e_n]}$$

In this formula, there is an identification of $[e_0, \dots, e_{i-1}, e_{i+1}, \dots, e_n]$ with Δ^{n-1}, preserving the ordering of vertices, so that $\sigma|_{[e_0,\dots,e_{i-1},e_{i+1},\dots,e_n]}$ is regarded as a singular $(n-1)$-simplex $\Delta^{n-1} \longrightarrow X$.

Remark To define ∂_0, we have to define $C_{-1}(X)$ as the free abelian group with basis the empty set. So C_{-1} is the trivial group and then ∂_0 is the trivial homomorphism.

Lemma 5.13 *The composition $\partial_n \circ \partial_{n+1} : C_{n+1}(X) \longrightarrow C_{n-1}(X)$ is zero, $\forall n \in \mathbb{N}$.*

Proof. For $n = 0$, the lemma is trivial. We will prove the lemma in the case $n = 1$.

$$\begin{aligned}\partial_1(\partial_2(\sigma)) &= \partial_1(\sigma|_{[e_1,e_2]} - \sigma|_{[e_0,e_2]} + \sigma|_{[e_0,e_1]}) \\ &= \sigma|_{[e_2]} - \sigma|_{[e_1]} - \sigma|_{[e_2]} + \sigma|_{[e_0]} + \sigma|_{[e_1]} - \sigma|_{[e_0]} = 0\end{aligned}$$

Definition 5.14 *Let X be a topological space and $n \in \mathbb{N}$. We define the n-th singular homology group by*

$$H_n(X) := \ker(\partial_n)/Im(\partial_{n+1})$$

This is well defined since the preceeding lemma.

Remark Let X, Y be topological spaces and $f : X \longrightarrow Y$ a continuous map. Then f induces an homomorphism from $C_n(X)$ to $C_n(Y)$, $\forall n \in \mathbb{N}$, in the following way. For every singular n-simplex σ in X we define $f_\sharp(\sigma) := f \circ \sigma$, which is an n-simplex in Y. Then we can extend $f_\sharp$ to an homomorphism $C_n(X) \longrightarrow C_n(Y)$ by linearity.

Theorem 5.15 *A continuous map $f : X \longrightarrow Y$ between topological spaces induces an homomorphism $f_* : H_n(X) \longrightarrow H_n(Y)$, $\forall n \in \mathbb{N}$. Moreover, if Z is a topological space and $g : Y \longrightarrow Z$ is a continuous map, then $(g \circ f)_* = g_* \circ f_*$.*

Proof. For the proof, consult [1], chap. 2, p. 111. This come from the fact that $f_\sharp$ has the property $\partial_n \circ f_\sharp = f_\sharp \circ \partial_n$.

Proposition 5.16 *Let X be a nonempty and path-connected space. Then*

$$H_0(X) \cong \mathbb{Z}$$

Hence, for any space X, $H_0(X)$ is a direct sum of copies of $\mathbb{Z}$, one for each path-component of X.

Remark The proof of the proposition 5.16 can be seen in [1], chap. 2, p. 109. From this proposition we see that if X is a point, $H_0(X) \cong \mathbb{Z}$. To avoid this fact, we make the following definition.

Definition 5.17 *Let X be a topological space. We consider the projection $X \longrightarrow *$, where $*$ is a topological space made of one point. By the theorem 5.15, this induces an homomorphism*

$$H_n(X) \longrightarrow H_n(*)$$

for every $n \in \mathbb{N}$. We define the reduced singular homology group $\widetilde{H}_n$ as the kernel of this homomorphism.

Remark In fact, $\widetilde{H}_n(X)$ is the group which makes the sequence

$$0 \longrightarrow \widetilde{H}_n(X) \longrightarrow H_n(X) \longrightarrow H_n(*) \longrightarrow 0$$

short exact. Since $H_0(*) \cong \mathbb{Z}$ and $H_n(*) = 0$ for $n \geq 1$ we get that

$$\widetilde{H}_n(X) \cong H_n(X), \quad n \geq 1$$

and

$$\widetilde{H}_0(X) = 0$$

if X is path connected.

Remark With the same hypothesis as in the theorem 5.15, f induces an homomorphism $f_* : \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(Y)$ with the same properties as in the theorem.

Proposition 5.18 *Let X, Y be topological spaces and f, g be two maps from X to Y. If $f \simeq g$, then the two induced homomorphisms f_* and g_* are equal.*

Proof. The proof is not trivial. It can be read in [1], chap. 2, p. 112.

Corollary 5.19 *Let X, Y be topological spaces. If $X \simeq Y$, then*

$$\widetilde{H}_n(X) \cong \widetilde{H}_n(Y)$$

$\forall n \in \mathbb{N}$. *In particular, if X is contractible, then $\widetilde{H}_n(X) = 0$, $\forall n \in \mathbb{N}$.*

Proof. By hypothesis, there exists $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $g \circ f \simeq Id_X$ and $f \circ g \simeq Id_Y$. By the preceeding proposition we get that

$$(g \circ f)_* = Id_{\widetilde{H}_n(X)} \quad \text{and} \quad (f \circ g)_* = Id_{\widetilde{H}_n(Y)}$$

But since $(g \circ f)_* = g_* \circ f_*$ and $(f \circ g)_* = f_* \circ g_*$, we get that $g_* = (f_*)^{-1}$ and so $f_* : \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(Y)$ is an isomorphism.

Proposition 5.20 *Let X be a topological space and $A \subseteq X$ be a nonempty closed subspace that is a deformation retract of some neighborhood in X. Then we have a long exact sequence of reduced homology groups*

$$... \longrightarrow \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X/A) \longrightarrow \widetilde{H}_{n-1}(A) \xrightarrow{i_*} ...$$
$$... \longrightarrow \widetilde{H}_0(X/A) \longrightarrow 0$$

where i_*, j_* are the homomorphisms induced by the inclusion $i : A \hookrightarrow X$ and the quotient map $j : X \longrightarrow X/A$.

Remark The proof of the preceeding proposition can be seen in [1], chap. 2, p. 114. We arrive now to the principal result of this section, that will be usefull to the next section : the Hurewicz theorem. This result is proved in [1], chap. 4, p. 366.

Theorem 5.21 *(Hurewicz) Let X be a $(n-1)$-connected space, $n \geq 2$. Then $\widetilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong \widetilde{H}_n(X)$.*

5.3 The plus-construction

Definition 5.22 *A CW-complex is a topological space X so that $X = \bigcup_{n \in \mathbb{N}} X_n$ where :*

1. *X_0 is a discreet space ;*
2. *$\forall n > 0$, there exists a set of indices I_n and a collection of maps*

   $$\{f_\alpha : S^{n-1}_\alpha \longrightarrow X_{n-1} \mid \alpha \in I_n\}$$

   so that X_n is the quotient space $(X_{n-1} \amalg \coprod_{\alpha \in I_n} D^n_\alpha)/\sim$, where we define $f_\alpha(x) \sim x$, $\forall x \in \partial D^n_\alpha = S^{n-1}_\alpha$, $\forall \alpha \in I_n$;

3. A subset $A \subseteq X$ is open if and only if $A \cap X_n$ is open in X_n for every $n \in \mathbb{N}$.

Example A Δ-complex is in particular a CW-complex.

Definition 5.23 *A continuous map $f : X \longrightarrow Y$ between two CW-complex is called cellular if $f(X_n) \subseteq Y_n$, $\forall n \in \mathbb{N}$.*

Definition 5.24 *Let X, A be topological spaces and $f : A \longrightarrow X$ be a continuous map. We define the cone of A by*

$$C(A) := ([0,1] \times A)/\sim$$

where $(0,a) \sim (0,a')$, $\forall a, a' \in A$, and the mapping cone of f by

$$C(f) := (C(A) \amalg X)/\sim$$

where $(1,a) \sim f(a)$, $\forall a \in A$.

Examples

1. If $f : A \longrightarrow X$ is simply the inclusion of a subspace, then $C(f) \simeq X/A$.
2. If $f : S^{n-1} \longrightarrow D^n$ is the inclusion, then $C(f) \cong S^n$. Effectively, the cone $C(S^{n-1})$ is clearly homeomorphic to D^n. Furthermore
$$(D_1^n \amalg D_2^n)/(\partial D_1^n \sim \partial D_2^n) \cong S^n$$
3. If X is a CW-complex and $f : S^{n-1} \longrightarrow X$ is a cellular map, then $C(f)$ is the CW-complex $(D^n \amalg X)/\sim$, where $f(x) \sim x$, $\forall x \in \partial D^n = S^{n-1}$.
4. Extending the preceeding example, if the space X is a CW-complex and if $f_\alpha : S_\alpha^{n-1} \longrightarrow X$, $\alpha \in I$, are cellular maps, then
$$C(f) \cong \left(X \amalg \coprod_{\alpha \in I} D_\alpha^n \right) / \sim$$
where $f_\alpha(x) \sim x$, $\forall x \in \partial D_\alpha^n = S_\alpha^{n-1}$, $\forall \alpha \in I$.

Proposition 5.25 *Let X, A be topological spaces and $f : A \longrightarrow X$ be a continuous map. Then the sequence*

$$A \xrightarrow{f} X \longrightarrow C(f)$$

is a cofibration sequence. Moreover, the long exact sequence of this cofibration gives rise to a long exact sequence

$$... \longrightarrow \widetilde{H}_n(A) \xrightarrow{f_*} \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(C(f)) \longrightarrow \widetilde{H}_{n-1}(A) \xrightarrow{f_*} ...$$
$$... \longrightarrow \widetilde{H}_0(C(f)) \longrightarrow 0$$

Proof. For this result, consult [1], chap. 4, p. 460-462.

Lemma 5.26 *Let I be a set of indices and $X_\alpha, \alpha \in I$, be topological spaces. Then*

$$\widetilde{H}_n(\bigvee_{\alpha\in I} X_\alpha) \cong \bigoplus_{\alpha \in I} \widetilde{H}_n(X_\alpha)$$

for every $n \in \mathbb{N}$.

Proof. The proof can be seen in [1], chap. 2, p. 126. In fact, this is the wedge axiom of a reduced homology theory and the reduced singular homology is one such theory.

Lemma 5.27 *Let $i \in \mathbb{N}$ and I be a set of indices. Then*

$$\widetilde{H}_i(\bigvee_{\alpha\in I} S^n_\alpha)) = 0 \text{ if } i \neq n$$

and

$$\widetilde{H}_n(\bigvee_{\alpha\in I} S^n_\alpha) \cong \bigoplus_{\alpha\in I} \mathbb{Z}$$

Proof. As seen in example 2 above, we have a cofibration

$$S^{n-1} \hookrightarrow D^n \longrightarrow S^n$$

By the proposition 5.25, we get a short exact sequence

$$... \longrightarrow \widetilde{H}_i(D^n) \longrightarrow \widetilde{H}_i(S^n) \longrightarrow \widetilde{H}_{i-1}(S^{n-1}) \longrightarrow \widetilde{H}_{i-1}(D^n) \longrightarrow ...$$
$$... \longrightarrow \widetilde{H}_0(S^n) \longrightarrow 0$$

Since D^n is contractible, $\widetilde{H}_i(D^n) = 0$, $\forall i \in \mathbb{N}$. Then we get an isomorphism

$$\widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1})$$

$\forall i \in \mathbb{N}^*$. Thus we just need to prove the lemma in the case $n = 0$.

For $i \in \mathbb{N}$ and writing $S^0 = \{a, b\}$, we get directly from the definition that $C_i(S^0)$ is the free abelian group with basis composed of $\sigma_a : \Delta^i \longrightarrow a$ and $\sigma_b : \Delta^i \longrightarrow b$. Hence

$$C_i(S^0) \cong \mathbb{Z}\{a, b\}$$

Then the boundary maps are given by $\partial_i(\sigma_a) = \sum_{k=0}^{i}(-1)^k a$ and $\partial_i(\sigma_b) = \sum_{k=0}^{i}(-1)^k b$. In consequence, if i is odd, ∂_i is the trivial homomorphism and if i is even and $i \geq 2$, ∂_i is the identity. Therefore

$$\widetilde{H}_i(S^0) \cong H_i(S^0) = C_i(S^0)/C_i(S^0) = 0 \text{ if } i \text{ is odd}$$

and

$$\widetilde{H}_i(S^0) \cong H_i(S^0) = 0/0 = 0 \text{ if } i \text{ is even and } i \geq 2$$

For $i = 0$ we get $H_0(S^0) = C_0(S^0)/0 \cong C_0(S^0) \cong \mathbb{Z}\{a, b\}$. To find the reduced homology group we write the short exact sequence

$$0 \longrightarrow \widetilde{H}_0(S^0) \longrightarrow H_0(S^0) \cong \mathbb{Z}\{a, b\} \longrightarrow H_0(*) \cong \mathbb{Z} \longrightarrow 0$$

But the homomorphism $\mathbb{Z}\{a, b\} \longrightarrow \mathbb{Z}$ is given by $a \longmapsto 1$ and $b \longmapsto 1$ and so we get that the kernel of this homomorphism is $\mathbb{Z}\{a - b\} \cong \mathbb{Z}$.

Remark Now we arrive to the main theorem of this chapter, which will allow us to construct a topological space that will give the K-theory groups. In this theorem, we suppose that $\widetilde{H}_1(X) = 0$, which means in fact that $\pi_1(X)$ is a perfect group. Then in the corollary we will consider a perfect subgroup of $\pi_1(X)$.

Theorem 5.28 *Let X be a connected CW-complex so that $\widetilde{H}_1(X) = 0$. Then there exists a simply-connected CW-complex X^+ and a continuous map $f^+ : X \longrightarrow X^+$ inducing isomorphisms on all reduced homology groups.*

Proof. First we take for each generator of $\pi_1(X)$ a map $\varphi_\alpha : S^1 \longrightarrow X$. Then we form X' as the quotient space

$$X' = \left(X \amalg \coprod_{\alpha \in I} D^2_\alpha \right) / \sim$$

where $\varphi_\alpha(x) \sim x$, $\forall x \in \partial D^2_\alpha = S^1_\alpha$, $\forall \alpha \in I$. By the cellular approximation theorem (see [1], chap. 4, p. 349), we can assume that every φ_α is cellular, that is X' is a CW-complex. Since X is a CW-complex and is a subcomplex of X', the hypothesis of the proposition 5.20 are satisfied (see [1], appendix, p. 523). Then we get the long exact sequence

$$\begin{aligned} ... \longrightarrow \widetilde{H}_{i+1}(X'/X) \longrightarrow \widetilde{H}_i(X) \longrightarrow \widetilde{H}_i(X') \longrightarrow \widetilde{H}_i(X'/X) \longrightarrow ... \\ ... \longrightarrow \widetilde{H}_3(X'/X) \longrightarrow \widetilde{H}_2(X) \longrightarrow \widetilde{H}_2(X') \longrightarrow \widetilde{H}_2(X'/X) \longrightarrow \widetilde{H}_1(X) \longrightarrow ... \end{aligned}$$

By hypothesis, $\widetilde{H}_1(X) = 0$. Furthermore, since we have attached D^2_α to X to obtain X', we get that $X'/X \cong \bigvee_{\alpha \in I} S^2_\alpha$ and so lemma 5.26 gives

$$\widetilde{H}_i(X'/X) \cong \widetilde{H}_i(\bigvee_{\alpha \in I} S^2_\alpha) \cong \bigoplus_{\alpha \in I} \widetilde{H}_i(S^2_\alpha)$$

Hence we get $\widetilde{H}_i(X'/X) = 0$ if $i \neq 2$ and $\widetilde{H}_2(X'/X) \cong \bigoplus_{\alpha \in I} \mathbb{Z}$ by lemma 5.27. In consequence we have from the long exact sequence that

$$\widetilde{H}_i(X') \cong \widetilde{H}_i(X) \text{ if } i \neq 2$$

Since $\bigoplus_{\alpha\in i}\mathbb{Z}$ is a free abelian group, we have that the short exact sequence

$$0 \longrightarrow \widetilde{H}_2(X) \longrightarrow \widetilde{H}_2(X') \longrightarrow \widetilde{H}_2(X'/X) \longrightarrow 0$$

splits and thus from lemma 3.9

$$\widetilde{H}_2(X') \cong \widetilde{H}_2(X) \oplus \bigoplus_{\alpha\in I}\mathbb{Z}$$

From the construction of X' we have that $\pi_1(X') = 0$. Then by the Hurewicz theorem

$$\pi_2(X') \cong \widetilde{H}_2(X') \cong \widetilde{H}_2(X) \oplus \bigoplus_{\alpha\in I}\mathbb{Z}$$

Then taking generators for $\widetilde{H}_2(X'/X)$ they correspond by the isomorphism to elements $[\psi_\alpha] \in \pi_2(X')$, $\alpha \in I$. We note X^+ the quotient space

$$X^+ = \left(X' \amalg \coprod_{\alpha\in I} D^3_\alpha\right) / \sim$$

where $\psi_\alpha(x) \sim x$, $\forall x \in \partial D^3_\alpha = S^2_\alpha$, $\forall \alpha \in I$. By the cellular approximation theorem, we can again assume that every ψ_α is cellular, that is X^+ is a CW-complex.

By the definition of X^+ and the example 3 above, we get that

$$\bigvee_{\alpha\in I} S^2_\alpha \overset{\vee\psi_\alpha}{\longrightarrow} X' \longrightarrow X^+$$

is a cofibration sequence. Then by proposition 5.25 we get the long exact sequence

$$\begin{aligned} ... \longrightarrow \widetilde{H}_i(\bigvee_{\alpha\in I} S^2_\alpha) \longrightarrow \widetilde{H}_i(X') \longrightarrow \widetilde{H}_i(X^+) \longrightarrow \widetilde{H}_{i-1}(\bigvee_{\alpha\in I} S^2_\alpha) \longrightarrow ... \\ ... \longrightarrow \widetilde{H}_3(\bigvee_{\alpha\in I} S^2_\alpha) \longrightarrow \widetilde{H}_3(X') \longrightarrow \widetilde{H}_3(X^+) \longrightarrow \widetilde{H}_2(\bigvee_{\alpha\in I} S^2_\alpha) \longrightarrow \\ \longrightarrow \widetilde{H}_2(X') \longrightarrow \widetilde{H}_2(X^+) \longrightarrow ... \longrightarrow \widetilde{H}_0(X^+) \end{aligned}$$

Since lemma 5.27 we get $\widetilde{H}_i(\bigvee_{\alpha\in I} S^2_\alpha) = 0$ if $i \neq 2$ and $\widetilde{H}_2(\bigvee_{\alpha\in I} S^2_\alpha) \cong \bigoplus_{\alpha\in I}\mathbb{Z}$. In consequence we have from the long exact sequence that

$$\widetilde{H}_i(X^+) \cong \widetilde{H}_i(X') \cong \widetilde{H}_i(X) \text{ if } i \neq 2, 3$$

and that

$$0 \longrightarrow \widetilde{H}_3(X') \longrightarrow \widetilde{H}_3(X^+) \longrightarrow \bigoplus_{\alpha\in I}\mathbb{Z} \overset{(\vee\psi_\alpha)_*}{\longrightarrow} \widetilde{H}_2(X') \longrightarrow \widetilde{H}_2(X^+) \longrightarrow 0$$

is split exact. Given an element $[f] \in \pi_2(\bigvee_{\alpha \in I} S^2_\alpha)$, we get an element in $\pi_2(X')$ by composing

$$S^2 \xrightarrow{f} \bigvee_{\alpha \in I} S^2_\alpha \xrightarrow{\vee \psi_\alpha} X'$$

Since $\pi_2(\bigvee_{\alpha \in I} S^2_\alpha) \cong \widetilde{H}_2(\bigvee_{\alpha \in I} S^2_\alpha) \cong \bigoplus_{\alpha \in I} \mathbb{Z}$ by the Hurewicz theorem, the generators of $\pi_2(\bigvee_{\alpha \in I} S^2_\alpha)$ are the equivalence classes of the maps

$$S^2 \xrightarrow{Id_{S^2}} S^2_\alpha \subseteq \bigvee_{\alpha \in I} S^2_\alpha$$

for $\alpha \in I$. Thus the image of those generators in $\pi_2(X')$ are in fact the ψ_α, $\alpha \in I$. Then the composition

$$\begin{array}{ccccc} \widetilde{H}_2(\bigoplus_\alpha \mathbb{Z}) & \xrightarrow{(\vee\psi_\alpha)_*} & \widetilde{H}_2(X') & \xrightarrow{\cong} & \widetilde{H}_2(X) \oplus \widetilde{H}_2(\bigvee_\alpha S^2_\alpha) \\ \cong \uparrow & & \cong \uparrow & & \\ \pi_2(\bigvee_\alpha S^2_\alpha) & \longrightarrow & \pi_2(X') & & \end{array}$$

send $\widetilde{H}_2(\bigoplus_{\alpha \in I} \mathbb{Z})$ onto the corresponding factor $\widetilde{H}_2(\bigvee_{\alpha \in I} S^2_\alpha)$ in $\widetilde{H}_2(X')$ via $(\vee\psi_\alpha)_*$. Finally, the long exact sequence of the cofibration

$$0 \longrightarrow \widetilde{H}_3(X') \longrightarrow \widetilde{H}_3(X^+) \longrightarrow \widetilde{H}_2(\bigoplus_{\alpha \in I} \mathbb{Z}) \xrightarrow{\vee\psi_\alpha}$$
$$\xrightarrow{\vee\psi_\alpha} \widetilde{H}_2(X') \cong \widetilde{H}_2(X) \oplus \widetilde{H}_2(\bigvee_{\alpha \in I} S^2_\alpha) \longrightarrow \widetilde{H}_2(X^+) \longrightarrow 0$$

gives $\widetilde{H}_3(X^+) \cong H_3(X') \cong \widetilde{H}_3(X)$ and $\widetilde{H}_2(X^+) \cong \widetilde{H}_2(X)$. By construction, $\pi_1(X^+) = \pi_1(X') = 0$ and so the theorem is proved.

Corollary 5.29 *Let X be a connected CW-complex. Then for every perfect subgroup H of $\pi_1(X)$ there is a connected CW-complex X^+ so that $\pi_1(X^+) \cong \pi_1(X)/H$ and $H_n(X^+) \cong H_n(X)$, $\forall n \in \mathbb{N}$.*

We call X^+ the plus-construction of X with respect to the perfect subgroup H.

Proof. By the classification theorem of covering spaces, there is a covering space $p : \widetilde{X} \longrightarrow X$ so that $\pi_1(\widetilde{X}) \cong H$. By theorem 5.28, there is a simply-connected CW-complex $\widetilde{X}^+$ and a map $f^+ : \widetilde{X} \longrightarrow \widetilde{X}^+$ so that $\widetilde{H}_i(\widetilde{X}^+) \cong \widetilde{H}_i(\widetilde{X})$ via f^+_*. We define

$$M_p := (\widetilde{X} \times [0,1] \amalg X)/\sim$$

where $(\widetilde{x}, 1) \sim p(\widetilde{x})$, $\forall \widetilde{x} \in \widetilde{X}$, the mapping cylinder of p. Then we define

$$X^+ := (M_p \amalg \widetilde{X}^+)/\sim$$

where $(\widetilde{x}, 0) \sim f^+(\widetilde{x})$, $\forall \widetilde{x} \in \widetilde{X}$. By the van Kampen theorem, we get that

$$\pi_1(M_p)/\pi_1(\widetilde{X}) \cong \pi_1(X^+)$$

But since $M_p \simeq X$ we get $\pi_1(M_p) \cong \pi_1(X)$ and so

$$\pi_1(X^+) \cong \pi_1(X)/\pi_1(\widetilde{X}) \cong \pi_1(X)/H$$

Clearly, $X^+/M_p \cong \widetilde{X}^+/\widetilde{X}$. Then for $n \in \mathbb{N}$,

$$\widetilde{H}_n(X^+/M_p) \cong \widetilde{H}_n(\widetilde{X}^+/\widetilde{X}) \cong 0$$

since $\widetilde{H}_n(\widetilde{X}^+) \cong \widetilde{H}_n(\widetilde{X})$ by the theorem 5.28. By the proposition 5.20, we get the long exact sequence

$$0 \longrightarrow \widetilde{H}_n(M_p) \longrightarrow \widetilde{H}_n(X^+) \longrightarrow \widetilde{H}_n(X^+/M_p) = 0$$

for every $n \in \mathbb{N}$. Then

$$\widetilde{H}_n(X^+) \cong \widetilde{H}_n(M_p) \cong \widetilde{H}_n(X)$$

since $M_p \simeq X$.

Definition 5.30 *(Quillen) Let A be a ring. We define the K-theory groups by*

$$K_i(A) := \pi_i(BGL(A)^+)$$

for $i \in \mathbb{N}^*$, where the plus-construction is given with respect to the perfect subgroup $E(A) \subseteq GL(A)$ ($\cong \pi_1(BGL(A))$ by proposition 5.9).

Proposition 5.31 *Milnor's $K_1(A)$ defined in chapter 3 is isomorphic to Quillen's $K_1(A)$.*

Proof. We denote Milnor's $K_1(A)$ by $K_1^M(A)$ and Quillen's by $K_1^Q(A)$. We have from the proposition 5.9 that $\pi_1(BGL(A)) \cong GL(A)$. Furthermore, the definition of $K_1^Q(A)$ and the corollary 5.29 give

$$K_1^Q(A) = \pi_1(BGL(A)^+) \cong \pi_1(BGL(A))/E(A) \cong GL(A)/E(A) = K_1^M(A)$$

Remark We have that the definition 5.30 for $K_2(A)$ coincides also with the $K_2(A)$ that we have defined in the preceeding chapter.

Moreover, $K_i(-)$ is a covariant functor from the category of rings and homomorphisms of rings to the category of abelian groups and homomorphisms of groups.

Conclusion

As a conclusion, I will say that algebraic K-theory is a huge and interesting subjet. Given an ideal I of a ring A, we can also define relatives K-theory groups $K_i(A, I)$. In the same way, we can define such groups for a category.

In addition, there is also a topological K-theory, that is in fact born before algebraic K-theory and has inspired it. There is obviously a link between them.

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